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## Diffusion algebras

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### Abstract

We define the notion of ‘diffusion algebras’. They are quadratic Poincaré–Birkhoff–Witt algebras which are useful in order to find exact expressions for the probability distributions of stationary states appearing in one-dimensional stochastic processes with exclusion. One considers processes in which one has  $N$  species, the number of particles of each species being conserved. All diffusion algebras are obtained. The known examples already used in applications are special cases in our classification. To help the reader interested in physical problems, the cases  $N = 3$  and  $4$  are listed separately.

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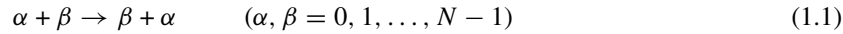
### 1. Introduction

One-dimensional stochastic processes with random-sequential updating have stationary states (far away from equilibrium) with very interesting physical properties. One observes phase transitions about which still little is known. As opposed to equilibrium states where the phase transitions are essentially a bulk phenomenon, in stationary states, the boundary conditions play an essential role [1]. One observes bulk induced phase transitions if one looks to problems on a ring [2], boundary induced phase transitions in the case of open systems [3], or a combination of both [4]. When dealing with phase transitions, one is interested in exact expressions for the relevant physical quantities like the current densities (which are zero for equilibrium problems) and various correlation functions. A major step in this direction was achieved when it was understood that in certain cases one can use the so-called ‘matrix product states’ approach [1,5]. In our opinion, from a mathematical point of view, this approach was loosely defined (for an illustration see appendix A). The aim of this paper is to improve upon this situation and, as a bonus, to give a large number of processes where one can find ‘matrix product states’. We define a class of quadratic algebras to which we have coined the name ‘diffusion algebras’.

We also show how to construct all of them. We believe that in this way one can bring some mathematical ‘beauty’ in what was not, up to now, a systematic approach.

We start with the physical problem in order to explain the motivation of our mathematical work.

In this paper we consider a restricted class of processes in which we take  $N$  species of particles with  $N - 1$  conservation laws in the bulk. We take a one-dimensional lattice with  $L$  sites and assume that one has only nearest-neighbour interaction with exclusion (there can be only one particle on a given site). In the time interval  $dt$  only the following processes are allowed in the bulk:



with the probability  $g_{\alpha\beta} dt$  ( $g_{\alpha\beta} \geq 0$ ). Equation (1.1) is a symbolic equation used by physicists. Its meaning is that the particles  $\alpha$  and  $\beta$  on successive sites exchange their places. Obviously the number of particles  $n_\alpha$  of each species  $\alpha$  are conserved ( $\sum_{\alpha=0}^{N-1} n_\alpha = L$ ). One is interested in the probability distribution  $P(\alpha_1, \alpha_2, \dots, \alpha_L)$ , ( $\alpha_k = 0, 1, \dots, N - 1$ ) for the stationary state. In order to obtain it, in the matrix product approach, one considers  $N$  matrices  $\mathcal{D}_\alpha$  and  $N$  matrices  $X_\alpha$  acting in an auxiliary vector space and satisfying the following relations [6]:

$$\begin{aligned} g_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta - g_{\beta\alpha} \mathcal{D}_\beta \mathcal{D}_\alpha &= \frac{1}{2} \{X_\beta, \mathcal{D}_\alpha\} - \frac{1}{2} \{X_\alpha, \mathcal{D}_\beta\} \\ [\mathcal{D}_\alpha, X_\beta] &= -[\mathcal{D}_\beta, X_\alpha]. \end{aligned} \quad (1.2)$$

In equations (1.2)  $\{.,.\}$  represent anti-commutators.

If one considers processes on a ring (periodic boundary condition), the un-normalized probability distribution has the following expression [7]:

$$P(\alpha_1, \alpha_2, \dots, \alpha_L) = \text{Tr}(\mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_2} \dots \mathcal{D}_{\alpha_L}). \quad (1.3)$$

Notice that the matrices  $X_\alpha$  do not appear in equation (1.3). The expression of the probability distribution is special in at least two ways. As a consequence of the conservation laws, equation (1.3) connects only monomials with the same numbers  $n_0, n_1, \dots, n_{N-1}$  of generators  $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{N-1}$ . This implies that one can use different matrices for monomials for different values of the set  $n_0, n_1, \dots, n_{N-1}$ . This observation is important since for example, a given infinite-dimensional representation can have a finite trace for certain class of monomials but can diverge for another class of monomials. In order to use equation (1.3), for the latter one can use a different representation which for example, is traceless for the first class of monomials but has a finite trace for the second class. This problem will be explained in detail in [8].

On the other hand if different representations can be used for all monomials one has the remarkable property that up to a factor (the expression (1.3) does not contain a normalization factor), the traces are independent on the representations one uses. For concrete calculations one therefore takes the representation with the smallest dimension.

If one considers open systems, the bulk processes have to be completed by boundary processes (they break the conservation laws). On the left side of the chain (site 1) and on the right side of the chain (site  $L$ ) we assume that in the time interval  $dt$ , the particle  $\alpha$  is replaced by the particle  $\beta$ :



with the probabilities

$$L_\beta^\alpha dt \quad \text{respectively} \quad R_\beta^\alpha dt. \quad (1.5)$$

The matrices appearing in (1.5) are intensity matrices [9] for which the diagonal elements are given by the non-diagonal ones

$$L_\alpha^\alpha = - \sum_{\beta \neq \alpha} L_\beta^\alpha \quad R_\alpha^\alpha = - \sum_{\beta \neq \alpha} R_\beta^\alpha. \tag{1.6}$$

The non-diagonal matrix elements are non-negative.

For open systems, the un-normalized probability distribution functions are given by a matrix element in the auxiliary vector space

$$P(\alpha_1, \alpha_2, \dots, \alpha_L) = \langle 0 | \mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_2} \dots \mathcal{D}_{\alpha_L} | 0 \rangle \tag{1.7}$$

where the bra (ket) state  $\langle 0 |$  (respectively  $| 0 \rangle$ ) are given by the following conditions:

$$\langle 0 | (L_\alpha^\beta \mathcal{D}_\beta + X_\alpha) = 0 \quad (R_\alpha^\beta \mathcal{D}_\beta - X_\alpha) | 0 \rangle = 0. \tag{1.8}$$

The expressions (1.3) and (1.7) for the probability distributions are of little use unless one finds matrices which satisfy the conditions (1.2) and have a trace, or one finds matrices which satisfy (1.2) and have the property (1.8). In a very nice paper [10] it was shown that for arbitrary bulk and boundary rates one can in principle construct infinite-dimensional matrices which satisfy the conditions (1.2) and (1.8) (if they also have finite traces is unclear). It is however practically impossible to obtain explicit expressions for these matrices. It is also not clear which supplementary relations besides (1.2) they satisfy. In other words, the algebraic structure behind the relations (1.2) is obscure.

In this paper we adopt a different approach. We start by searching for quadratic algebras which are of PBW type (this notion is explained in the beginning of section 2) with a structure which is understood and look for those bulk rates for which a simplified version of the relations (1.2) exist. Then we look to boundary conditions compatible with the simplified version of equation (1.8). We make the ansatz

$$X_\alpha = x_\alpha e \tag{1.9}$$

where  $e$  acts as a unit element on  $\mathcal{D}_\alpha$

$$e \mathcal{D}_\alpha = \mathcal{D}_\alpha e = \mathcal{D}_\alpha \tag{1.10}$$

and  $x_\alpha$  are  $c$ -numbers. With this ansatz, instead of equation (1.2) one obtains  $N(N - 1)/2$  relations for the matrices  $\mathcal{D}_\alpha$ :

$$g_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta - g_{\beta\alpha} \mathcal{D}_\beta \mathcal{D}_\alpha = x_\beta \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\beta \tag{1.11}$$

and instead of equation (1.8) one obtains

$$\langle 0 | (L_\alpha^\beta \mathcal{D}_\beta + x_\alpha e) = 0 \quad (R_\alpha^\beta \mathcal{D}_\beta - x_\alpha e) | 0 \rangle = 0. \tag{1.12}$$

Taking into account the relations (1.6), one has

$$\sum_{\alpha=0}^{N-1} x_\alpha = 0.$$

The consequence of the ansatz (1.9) will be that we will not be able to find expressions for the probability distributions for arbitrary bulk and boundary rates. Further limitations on the possible bulk rates will appear when we ask for the relations (1.11) to define quadratic algebras of PBW type. This implies that the ordered monomials

$$\mathcal{D}_{N-1}^{n_0} \mathcal{D}_{N-2}^{n_1} \dots \mathcal{D}_0^{n_{N-1}} \tag{1.13}$$

form a basis in the algebra. Algebras of this type will be called ‘diffusion algebras’.

Once the algebras are known, we can ask if they are also useful for applications to stochastic processes. First one has to find for which algebras one can choose all the  $g_{\alpha\beta}$  non-negative.

Next, since the relation (1.11) gives recurrence relations among traces of different monomials, one has to find in which cases one has not only the trivial solution for these recurrence relations (all the traces vanish). These cases can be used to study stochastic processes on a ring. Finally, one has to find for which boundary matrices one can find representations for which the conditions (1.12) are satisfied. This is not a trivial exercise. For the cases for which one finds solutions, one can get then the expression of the probability distribution for the open system.

We would like to give a supplementary argument which motivated us to look for PBW algebras. Interesting physics appears when one has only infinite-dimensional representations either for the ring problem or for the open system [1, 2, 5]. The PBW algebras have at least one infinite-dimensional representation (the regular one), which does not mean that, in general, they do not also have finite-dimensional ones.

In this paper we concentrate on the diffusion algebras only. In a sequel [8] we will discuss in more detail the representation theory. We will also not look for physical applications except for stressing the cases when the rates  $g_{\alpha\beta}$  cannot be chosen non-negative.

In section 2 we will show that in order to have a diffusion algebra, the  $g_{\alpha\beta}$  and  $x_\alpha$  have to satisfy certain identities which are the equivalent of the Jacobi identities for the structure constants in the case of Lie algebras<sup>4</sup>. It is remarkable that, as we are going to show in the next sections, we are able to find all the solutions of these identities. Inspecting the equations (1.11) one notices that if some of the  $x_\alpha$  are not zero they can be rescaled in the definition of the generators  $\mathcal{D}_\alpha$ . The number of non-vanishing  $x_\alpha$  will play an important role in the classification of diffusion algebras.

In section 3 we consider the case  $N = 3$  corresponding to the three species problem (the case  $N = 2$  is well known [11]). The case  $N = 3$  is not only interesting on its own but is relevant for understanding the case for arbitrary  $N$  described in section 4. We are going to recapture all the known examples [2, 9, 12, 13] which are interesting for applications and get a few new algebras which are interesting on their own.

We would like to mention that the case  $N = 3$  was approached previously from two other points of view. In [9], one has looked directly to the open system, and classified the type of boundary matrices appearing in equation (1.12). This is possible since the boundary matrices are intensity matrices which, as explained in [9], have special properties. Next, one has looked for bulk rates (see equation (1.11)) compatible with the boundary conditions. The problem on the ring was considered in [12]. Here one has asked which relations (1.11) are compatible with the trace operation and one has given the smallest representations (this was enough for applications).

In appendix A we present an instructive different (although equivalent) approach to obtain part of the diffusion algebras for  $N = 3$ . We also show a natural way to define a quotient of one of the algebras.

In section 4 we consider the case of  $N$  generators. First we give seven series of algebras and then present a theorem which allows one to find all the diffusion algebras of PBW type. The proof of this theorem would imply a long discussion and would not fit in this paper which is also aimed at physicists who are not necessarily interested in combinatorics. We have therefore decided to publish it separately. Some algebras are not suitable for applications to stochastic processes (they are not compatible with positive rates).

In order to help the reader who is interested in applications and not in mathematics, in appendix B we list the diffusion algebras with positive rates for  $N = 4$ . In section 5 several physically meaningful generalizations of the diffusion algebras are discussed. A possible connection between diffusion algebras and quantum Lie algebras is also pointed out.

<sup>4</sup> Correctly speaking, we mean universal enveloping algebras of Lie algebras.

### 2. Diamond conditions

We shall search for PBW-type associative algebras which are generated by the unit  $e$  and the elements  $\mathcal{D}_\alpha, \alpha = 0, 1, \dots, N - 1$  satisfying  $N(N - 1)/2$  quadratic-linear relations given by (1.11).

The PBW property (see, e.g., section 3 of the [14]) implies that given a set of generators  $\{\mathcal{D}_\alpha\}$ , one can express any element of the algebra as a linear combination of ordered monomials in  $\mathcal{D}_\alpha$ . Furthermore, all the ordered monomials are assumed to be linearly independent<sup>5</sup>.

For concreteness let us fix an alphabetic order

$$\mathcal{D}_\beta > \mathcal{D}_\alpha \quad \text{if } \beta > \alpha. \tag{2.1}$$

Then, a linear basis for the PBW-type algebra is given by the unit  $e$  and the set of monomials

$$\mathcal{D}_{\alpha_1}^{n_1} \mathcal{D}_{\alpha_2}^{n_2} \dots \mathcal{D}_{\alpha_k}^{n_k} \quad k = 1, 2, \dots \tag{2.2}$$

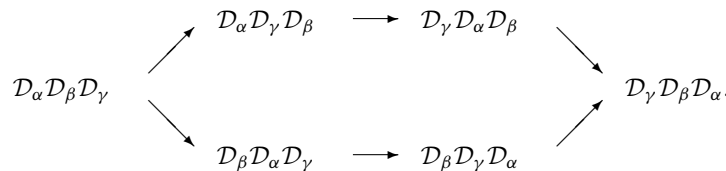
where  $\alpha_1 > \alpha_2 > \dots > \alpha_k$  and  $n_1, n_2, \dots, n_k$  are arbitrary positive integers.

Imposing the PBW condition for the diffusion algebra (1.11) we first demand

$$g_{\alpha\beta} \neq 0 \quad \forall \alpha < \beta \tag{2.3}$$

in order to be able to express any polynomial in  $\mathcal{D}_\alpha$  as a linear combination of the basic monomials (2.2).

Next, using (1.11) one can reorder any cubic monomial  $\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \rightarrow \mathcal{D}_\gamma \mathcal{D}_\beta \mathcal{D}_\alpha$ , where  $\alpha < \beta < \gamma$  in two different ways:



Demanding the coincidence of the resulting expressions for  $\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma$  in terms of ordered monomials one obtains the relation

$$\begin{aligned} &x_\alpha g_{\gamma\beta} (\Lambda_{\alpha\beta\gamma} + \Lambda_{\gamma\beta}) \mathcal{D}_\gamma \mathcal{D}_\beta + x_\beta g_{\gamma\alpha} (\Lambda_{\alpha\beta\gamma} + \Lambda_{\alpha\gamma}) \mathcal{D}_\gamma \mathcal{D}_\alpha \\ &+ x_\gamma g_{\beta\alpha} (\Lambda_{\alpha\beta\gamma} + \Lambda_{\beta\alpha}) \mathcal{D}_\beta \mathcal{D}_\alpha + x_\alpha x_\beta (g_{\gamma\alpha} - g_{\gamma\beta} - \Lambda_{\alpha\beta\gamma}) \mathcal{D}_\gamma \\ &+ x_\alpha x_\gamma (g_{\gamma\beta} - g_{\beta\alpha}) \mathcal{D}_\beta + x_\alpha x_\beta (g_{\beta\alpha} - g_{\gamma\alpha} + \Lambda_{\alpha\beta\gamma}) \mathcal{D}_\alpha = 0 \end{aligned} \tag{2.4}$$

which results in the following six conditions for the  $g_{\alpha\beta}$  and the  $x_\alpha$

$$x_\alpha g_{\gamma\beta} (\Lambda_{\alpha\gamma} - \Lambda_{\alpha\beta}) = 0 \tag{2.5}$$

$$x_\beta g_{\gamma\alpha} (\Lambda_{\beta\gamma} + \Lambda_{\alpha\beta}) = 0 \tag{2.6}$$

$$x_\gamma g_{\beta\alpha} (\Lambda_{\alpha\gamma} - \Lambda_{\beta\gamma}) = 0 \tag{2.7}$$

$$x_\beta x_\gamma (\Lambda_{\beta\gamma} + g_{\alpha\beta} - g_{\alpha\gamma}) = 0 \tag{2.8}$$

$$x_\alpha x_\gamma (g_{\beta\alpha} - g_{\gamma\beta}) = 0 \tag{2.9}$$

$$x_\alpha x_\beta (\Lambda_{\alpha\beta} + g_{\beta\gamma} - g_{\alpha\gamma}) = 0 \quad \forall \alpha < \beta < \gamma. \tag{2.10}$$

In equations (2.4)–(2.10) we have introduced the notation

$$\Lambda_{\alpha\beta} := g_{\alpha\beta} - g_{\beta\alpha} \quad \Lambda_{\alpha\beta\gamma} := \Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}. \tag{2.11}$$

One can show that the conditions (2.5)–(2.10) are necessary in order to avoid linear dependences between ordered quadratic (or even first order) monomials in  $\mathcal{D}_\alpha, \mathcal{D}_\beta$  and  $\mathcal{D}_\gamma$ .

<sup>5</sup> The term PBW is due to Poincaré *et al* [15] who describe a linear basis in a universal enveloping algebra.

One can improve this result applying the diamond lemma [14] to our concrete case. Namely, the algebra (1.11) possesses the PBW property iff the conditions (2.5)–(2.10) are fulfilled. We will refer relations (2.5)–(2.10) as ‘diamond conditions’ in what follows.

In the next sections we are going to find the solutions of the diamond conditions first for the case  $N = 3$  and then in the general case.

### 3. Classification of diffusion algebras with 3 generators

The classification of diffusion algebras generated by an arbitrary number  $N$  of elements  $\mathcal{D}_\alpha$  proceeds as follows.

First, one notes that any subset of  $k < N$  elements  $\mathcal{D}_\alpha$  generates a subalgebra of (1.11) which is again a diffusion algebra. So, it looks natural to begin with minimal size subalgebras produced by three generators (for the case of two generators one does not have any nontrivial diamond conditions to solve). Then, a closer inspection of the diamond relations (2.5)–(2.10) shows that they would be fulfilled for the whole algebra (1.11) provided that they are satisfied for all the minimal size subalgebras.

So, we shall start by classifying diffusion algebras generated by three elements  $\mathcal{D}_\alpha, \mathcal{D}_\beta, \mathcal{D}_\gamma$ ,  $\alpha < \beta < \gamma$ . It is natural to fix the values of indices as  $\alpha = 0, \beta = 1, \gamma = 2$  (this choice is adopted in appendix A). Here however we will not assign concrete values to the indices  $\alpha, \beta$  and  $\gamma$  keeping in mind that for general  $N > 3$   $\alpha < \beta < \gamma$  may denote any triple of indices from the set  $0, 1, \dots, N - 1$ .

Depending on how many of the parameters  $x_\alpha, x_\beta, x_\gamma$  take nonzero values the classification falls into four cases. Namely, when all three parameters  $x_\alpha, x_\beta$  and  $x_\gamma$  are nonzero we obtain the algebras of type *A*. If one of the  $x$  is zero and the remaining two are not equal to zero we have the algebras of type *B*. The algebras with only one nonzero parameter  $x$  and those with all  $x$  zero are called algebras of type *C* and respectively *D*.

Now we consider in details the algebras of type *A–D*.

*Case A.* All  $x_\alpha, x_\beta$  and  $x_\gamma$  are nonzero.

Equations (2.8)–(2.10) give us two constraints

$$g_{\beta\alpha} = g_{\gamma\beta} = g_{\alpha\beta} + g_{\beta\gamma} - g_{\alpha\gamma}. \quad (3.1)$$

Then, there are two possibilities.

(1) If  $g_{\beta\alpha} = g_{\gamma\beta} \neq 0$ , the equations (2.5) and (2.7) give  $\Lambda_{\alpha\gamma} = \Lambda_{\alpha\beta} = \Lambda_{\beta\gamma}$  whereof one obtains

$$g_{\gamma\alpha} = g_{\beta\gamma} = g_{\alpha\beta}. \quad (3.2)$$

In view of (2.3) one has  $g_{\gamma\alpha} \neq 0$  and, therefore, from equation (2.6) one obtains

$$\Lambda_{\beta\gamma} = -\Lambda_{\alpha\beta}. \quad (3.3)$$

Finally, from (3.1), (3.2) and (3.3) one concludes

$$g_{ij} = g \neq 0 \quad \forall i, j \in \{\alpha, \beta, \gamma\}. \quad (3.4)$$

The corresponding  $A_I$ -type algebra is

$$A_I \quad g [\mathcal{D}_i, \mathcal{D}_j] = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad \forall i \neq j \in \{\alpha, \beta, \gamma\} \quad g \neq 0. \quad (3.5)$$

These are relations of Lie algebraic type. By rescaling the generators  $\mathcal{D}_i \rightarrow \frac{x_i}{g} E_i$  one can remove all the parameters from (3.5) and obtain

$$[E_i, E_j] = E_i - E_j.$$

(2) If  $g_{\beta\alpha} = g_{\gamma\beta} = 0$ , using equation (3.1) one transforms the only remaining nontrivial equation (2.6) as

$$g_{\gamma\alpha}(\Lambda_{\beta\gamma} + \Lambda_{\alpha\beta}) = g_{\gamma\alpha}(g_{\beta\gamma} + g_{\alpha\beta}) = g_{\gamma\alpha}g_{\alpha\gamma} = 0 \Rightarrow g_{\gamma\alpha} = 0.$$

The corresponding  $A_{II}$ -type algebra is

$$A_{II} \quad \begin{aligned} g_{ij}\mathcal{D}_i\mathcal{D}_j &= x_j\mathcal{D}_i - x_i\mathcal{D}_j & \forall i < j \in \{\alpha, \beta, \gamma\} \\ \text{where } g_{ij} &:= g_i - g_j & g_i \neq g_j \quad \forall i \neq j. \end{aligned} \quad (3.6)$$

The parameters  $g_i$  ( $i \in \{\alpha, \beta, \gamma\}$ ) introduced in equation (3.6) above are defined up to a common shift  $g_i \rightarrow g_i + c$ .

This algebra is invariant under the transformation

$$\mathcal{D}'_i = \mathcal{D}_i + \frac{x_i}{g_i - y} \quad g'_i = \frac{1}{y - g_i} \quad x'_i = \frac{x_i}{(y - g_i)^2}$$

where  $y$  is an arbitrary parameter. As explained in appendix A, the algebra (3.6) has a natural quotient  $[\mathcal{D}_\beta, \mathcal{D}_\gamma] = 0$ . We would like to mention that this algebra is already known [13].

Note that the algebras  $A_I$  and  $A_{II}$  can directly be extended to the case  $i, j = 0, 1, 2, \dots, N - 1$  for  $N > 3$  (they correspond to the algebras  $A_I(N)$  and  $A_{II}(N)$  discussed in section 4).

*Case B.* Among the coefficients  $x_\alpha, x_\beta$  and  $x_\gamma$  one is equal to zero.

Let  $x_\alpha, x_\gamma \neq 0$   $x_\beta = 0$ . In this case, equations (2.6), (2.8), (2.10) become trivial and equation (2.9) gives

$$g_{\beta\alpha} = g_{\gamma\beta}. \quad (3.7)$$

There are two ways to satisfy the remaining equations (2.5) and (2.7).

- (1) Equations (2.5) and (2.7) are satisfied if one chooses  $\Lambda_{\alpha\gamma} = \Lambda_{\alpha\beta} = \Lambda_{\beta\gamma} =: \Lambda$  which, in view of (3.7), leads to

$$g_{\alpha\beta} = g_{\beta\gamma} \quad g_{\gamma\alpha} = g_{\alpha\gamma} + g_{\beta\alpha} - g_{\alpha\beta}. \quad (3.8)$$

The corresponding algebra is

$$B^{(1)} \quad \begin{aligned} g_\beta \mathcal{D}_\alpha \mathcal{D}_\beta - (g_\beta - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\beta \\ g \mathcal{D}_\alpha \mathcal{D}_\gamma - (g - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha &= x_\gamma \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\gamma \\ g_\beta \mathcal{D}_\beta \mathcal{D}_\gamma - (g_\beta - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\beta &= x_\gamma \mathcal{D}_\beta \quad \forall g \quad g_\beta \neq 0. \end{aligned} \quad (3.9)$$

Here for sake of future convenience we have parametrized the bulk rates via  $g_\beta, g$  and  $\Lambda$ . The algebra (3.9) is also known [9, 12].

- (2) If  $g_{\gamma\beta} = g_{\beta\alpha} = 0$ , then equations (2.5), (2.7) are trivially satisfied and the algebra reads

$$B^{(2)} \quad \begin{aligned} g_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta &= -x_\alpha \mathcal{D}_\beta \\ g_{\alpha\gamma} \mathcal{D}_\alpha \mathcal{D}_\gamma - g_{\gamma\alpha} \mathcal{D}_\gamma \mathcal{D}_\alpha &= x_\gamma \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\gamma \\ g_{\beta\gamma} \mathcal{D}_\beta \mathcal{D}_\gamma &= x_\gamma \mathcal{D}_\beta \quad \forall g_{\alpha\beta}, g_{\alpha\gamma}, g_{\beta\gamma} \neq 0. \end{aligned} \quad (3.10)$$

This algebra can be found already in [9, 12]. Notice that if one takes  $g_\beta = \Lambda$  in equation (3.9) one obtains a special case of the algebra  $B^{(2)}$  (equation (3.10)).



We will not repeat the same considerations for the cases  $x_\alpha, x_\beta \neq 0, x_\gamma = 0$  and  $x_\beta, x_\gamma \neq 0, x_\alpha = 0$ . The resulting algebras are

$$\begin{aligned} g \mathcal{D}_\alpha \mathcal{D}_\beta - (g - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= x_\beta \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\beta \\ g_\gamma \mathcal{D}_\alpha \mathcal{D}_\gamma - (g_\gamma - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\gamma \\ (g_\gamma - \Lambda) \mathcal{D}_\beta \mathcal{D}_\gamma - g_\gamma \mathcal{D}_\gamma \mathcal{D}_\beta &= -x_\beta \mathcal{D}_\gamma \quad \forall g \neq 0 \quad g_\gamma \notin \{0, \Lambda\} \end{aligned} \quad (3.11)$$

$$\begin{aligned} (g_\alpha - \Lambda) \mathcal{D}_\alpha \mathcal{D}_\beta - g_\alpha \mathcal{D}_\beta \mathcal{D}_\alpha &= x_\beta \mathcal{D}_\alpha \\ g_\alpha \mathcal{D}_\alpha \mathcal{D}_\gamma - (g_\alpha - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha &= x_\gamma \mathcal{D}_\alpha \\ g \mathcal{D}_\beta \mathcal{D}_\gamma - (g - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\beta &= x_\gamma \mathcal{D}_\beta - x_\beta \mathcal{D}_\gamma \quad \forall g \neq 0 \quad g_\alpha \notin \{0, \Lambda\} \end{aligned} \quad (3.12)$$

$$\begin{aligned} g \mathcal{D}_\alpha \mathcal{D}_\beta - (g - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= x_\beta \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\beta \\ B^{(3)} \quad g_\gamma \mathcal{D}_\alpha \mathcal{D}_\gamma &= -x_\alpha \mathcal{D}_\gamma \\ (g_\gamma - \Lambda) \mathcal{D}_\beta \mathcal{D}_\gamma &= -x_\beta \mathcal{D}_\gamma \quad \forall g \neq 0 \quad g_\gamma \notin \{0, \Lambda\} \end{aligned} \quad (3.13)$$

$$\begin{aligned} (g_\alpha - \Lambda) \mathcal{D}_\alpha \mathcal{D}_\beta &= x_\beta \mathcal{D}_\alpha \\ B^{(4)} \quad g_\alpha \mathcal{D}_\alpha \mathcal{D}_\gamma &= x_\gamma \mathcal{D}_\alpha \\ g \mathcal{D}_\beta \mathcal{D}_\gamma - (g - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\beta &= x_\gamma \mathcal{D}_\beta - x_\beta \mathcal{D}_\gamma \quad \forall g \neq 0 \quad g_\alpha \notin \{0, \Lambda\}. \end{aligned} \quad (3.14)$$

The algebras (3.11) and (3.12) are just different presentations of the algebra  $B^{(1)}$  (one has to make the substitution  $\beta \leftrightarrow \gamma$  in equation (3.11) and  $\alpha \leftrightarrow \beta$  in equation (3.12)).

The relation between the algebras  $B^{(3)}$  and  $B^{(4)}$  is less trivial (therefore we keep them as different cases in classification). One can obtain the relations for the  $B^{(3)}$  algebra by inverting the order of all products in the  $B^{(4)}$  algebra (equations (3.14)), i.e. by reading the relations (3.14) from the right to the left and changing the signs of all  $x$ . That means, the algebras  $B^{(3)}$  and  $B^{(4)}$  describe mirror (left–right) symmetric physical processes.

The algebras  $B^{(3)}$  and  $B^{(4)}$  are completely different from  $B^{(2)}$ . Even the number of independent rates in cases  $B^{(3)}$  and  $B^{(4)}$  is not the same as in  $B^{(2)}$ .

*Case C.* Two of the coefficients  $x_\alpha, x_\beta$  and  $x_\gamma$  are equal to zero.

In this case only one of the equations (2.5)–(2.10) remains nontrivial and the analysis becomes straightforward. Below we present the relations for the type *C* algebras with  $x_\alpha \neq 0, x_\beta = x_\gamma = 0$ . The expressions for the cases  $x_\beta \neq 0, x_\alpha = x_\gamma = 0$  and  $x_\gamma \neq 0, x_\alpha = x_\beta = 0$  can be obtained by the substitutions  $\alpha \leftrightarrow \beta$  and, respectively,  $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$  in (3.15) and (3.16):

$$\begin{aligned} g_\beta \mathcal{D}_\alpha \mathcal{D}_\beta - (g_\beta - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\beta \\ C^{(1)} \quad g_\gamma \mathcal{D}_\alpha \mathcal{D}_\gamma - (g_\gamma - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\gamma \\ g_{\beta\gamma} \mathcal{D}_\beta \mathcal{D}_\gamma - g_{\gamma\beta} \mathcal{D}_\gamma \mathcal{D}_\beta &= 0 \quad \forall g_\beta, g_\gamma, g_{\beta\gamma} \neq 0 \end{aligned} \quad (3.15)$$

$$\begin{aligned} g_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta - g_{\beta\alpha} \mathcal{D}_\beta \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\beta \\ C^{(2)} \quad g_{\alpha\gamma} \mathcal{D}_\alpha \mathcal{D}_\gamma - g_{\gamma\alpha} \mathcal{D}_\gamma \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\gamma \\ g_{\beta\gamma} \mathcal{D}_\beta \mathcal{D}_\gamma &= 0 \quad \forall g_{\alpha\beta}, g_{\alpha\gamma}, g_{\beta\gamma} \neq 0. \end{aligned} \quad (3.16)$$

We observe that for  $g_{\gamma\beta} = 0$ , the algebra (3.15) is a special case of the algebra (3.16). For the sake of convenience (see section 4), we will not stress this observation any further.

Notice that in case  $\Lambda \neq 0$  shifting the generator  $\mathcal{D}_\alpha: \mathcal{D}_\alpha \rightarrow \mathcal{D}_\alpha - x_\alpha/\Lambda$ , in equation (3.15) one obtains

$$\begin{aligned} g_\beta \mathcal{D}_\alpha \mathcal{D}_\beta - (g_\beta - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= 0 \\ g_\gamma \mathcal{D}_\alpha \mathcal{D}_\gamma - (g_\gamma - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha &= 0 \\ g_{\beta\gamma} \mathcal{D}_\beta \mathcal{D}_\gamma - g_{\gamma\beta} \mathcal{D}_\gamma \mathcal{D}_\beta &= 0 \end{aligned} \quad (3.17)$$

which brings the  $C^{(1)}$  algebra to the subcase of a family of quantum hyperplanes (see case  $D$  below)<sup>6</sup>. If one thinks of applications to stochastic processes, the shift we made is not an innocent one since for  $g_{\beta\gamma} = g_{\gamma\beta}$ , the algebra (3.15) has representations with traces (for example, the one-dimensional representation given by the shift) whereas the algebra (3.17) has none.

A useful algebra (not of PBW type) is obtained if one takes not only  $g_{\gamma\beta} = 0$  but also  $g_{\beta\gamma} = 0$  in the algebra  $C^{(2)}$  given by equation (3.16). In this way one obtains the algebra used in [2].

*Case D.* All the coefficients  $x_\alpha$ ,  $x_\beta$  and  $x_\gamma$  are equal to zero.

In this case we obtain the algebra of Manin's quantum hyperplane [16] corresponding to multiparametric Drinfeld–Jimbo  $R$ -matrix [17]

$$D \quad g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_a \mathcal{D}_b = 0 \quad \forall a, b \in \{\alpha, \beta, \gamma\} \quad a < b \quad g_{ab} \neq 0. \quad (3.18)$$

A different point of view in understanding some of the algebras presented here is discussed in appendix A.

#### 4. Diffusion algebras with $N > 3$ generators

While classifying the PBW-type algebras (1.11) with more than 3 generators one meets the combinatorial problem of consistently combining several minimal subalgebras generated by triples  $\{\mathcal{D}_\alpha, \mathcal{D}_\beta, \mathcal{D}_\gamma\}$  (each of these subalgebras belonging to one of the types  $A$ – $D$  listed in section 3) to a larger algebra. In this section we shall first construct several basic series of diffusion algebras being extensions of the  $A_I$ ,  $A_{II}$ ,  $B^{(1)-(4)}$ ,  $C^{(1)}$  and  $D$ -type algebras from the previous section. The  $C^{(2)}$ -type triples will appear later on in our considerations. There is a deep reason behind our choice of starting first with the algebra  $C^{(1)}$  and taking into account the algebra  $C^{(2)}$  later. In this way one can more easily state the theorem presented at the end of this section which forms the central part of our work. A proof of this theorem will be given elsewhere.

In section 3 we have shown that the classification of diffusion algebras with  $N = 3$  generators depends essentially on the number of nonvanishing parameters  $x_\alpha$  in the ansatz (1.11). The same is true for general  $N$ . Therefore we shall split the set  $\{\alpha\}$  labelling different species of particles (= different generators  $\mathcal{D}_\alpha$ ) into two subsets  $\{\alpha\} = \{i\} \cup \{a\}$ . From now on we assign letters  $i, j, k$ , etc to the indices of the first subset and assume that  $x_i, x_j, x_k, \dots \neq 0$ . The indices of the second subset are denoted by letters  $a, b, c$ , etc and it is implied that  $x_a = x_b = x_c = \dots = 0$ . Let  $N_1$  and  $N_0$  denote the number of elements of the first and second subsets. Clearly,  $N_0 + N_1 = N$ —the total number of indices of both kinds.

We should stress however that  $N_1$ —the number of nonzero  $x$ —is the most noticeable but not the only relevant information for the classification. A supplementary information is given by a number of nonzero bulk rates  $g_{\alpha\beta}$  in the defining relations of the algebras (cf cases  $A_I$  and  $A_{II}$ ,  $B^{(1)}$  and  $B^{(2)}$ , or  $C^{(1)}$  and  $C^{(2)}$  from section 3) and the mutual arrangement of the indices  $\{i\}$  and  $\{a\}$  in the alphabetic order (see equation (2.1) and the definition of the algebras  $B^{(2)}$ ,  $B^{(3)}$  and  $B^{(4)}$  from section 3).

*Algebras of type A.* We shall start by considering the algebras with the number  $N_1$  of nonzero  $x$  not less than 3—we call them algebras of type  $A$ . Obviously, any such algebra contains a

<sup>6</sup> Conversely, using the linear shifts of generators  $\mathcal{D}'_i = \mathcal{D}_i + u_i$ ,  $\forall i \in \{\alpha, \beta, \gamma\}$  in the  $D$ -type algebra (3.18) and demanding the resulting relations to agree with the diffusion algebra ansatz (1.11) one recovers the  $C^{(1)}$ -type algebras only.

minimal subalgebra of type  $A_I$ , or  $A_{II}$ . Consequently, these algebras are naturally obtained by a sequence of consistent (in a sense of diamond conditions) extensions starting with the  $N = N_1 = 3$  algebras (3.5) or (3.6) and adding one new generator  $\mathcal{D}_\alpha$  at each step of the iteration.

It is suitable to begin the extension procedure with the generators whose indices lie in the subset  $\{i\}$ . At the first step one adds a fourth generator, say  $\mathcal{D}_l$ , to the triple  $\{\mathcal{D}_i, \mathcal{D}_j, \mathcal{D}_k\}$  (recall, once again, that  $x_i, x_j, x_k, x_l \neq 0$ ). The resulting algebra contains four minimal subalgebras of the types either  $A_I$ , or  $A_{II}$ . An easy check shows that it is possible to combine only triples of the same type. Continuing the extension procedure one finally obtains algebras with  $N_1$  generators for which all the minimal subalgebras are of the same type, either  $A_I$  or  $A_{II}$ . The defining relations for these two types of diffusion algebras— $A_I(N_1)$  and  $A_{II}(N_1)$ —are given by the equations (3.5) and, respectively (3.6), with  $i < j$  spanning the whole set  $\{i\}$ .

We continue the extension procedure adding to the algebras  $A_{I,II}(N_1)$ ,  $N_0$  new generators with their indices lying in the subset  $\{a\}$ . First, we add one generator, say  $\mathcal{D}_a$ , and take care that all the newly appeared triples  $\{\mathcal{D}_a, \mathcal{D}_i, \mathcal{D}_j\}$  belong to one of the algebras of type  $B$  (see equations (3.9), (3.10), (3.13) and (3.14)). Next, adding a second generator, say  $\mathcal{D}_b$ , we again require that all the triples  $\{\mathcal{D}_b, \mathcal{D}_i, \mathcal{D}_j\}$  are of type  $B$  and, moreover, demand that the triples  $\{\mathcal{D}_a, \mathcal{D}_b, \mathcal{D}_i\}$  are the generators of a  $C_I$ -type algebra (3.15). Adding new generators we also have to impose  $D$ -type algebraic relations for the triples  $\{\mathcal{D}_a, \mathcal{D}_b, \mathcal{D}_c\}$ . As a result, we obtain two different extensions for the algebra  $A_I(N_1)$

$$\begin{aligned}
 & g[\mathcal{D}_i, \mathcal{D}_j] = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad \forall i, j \in \{i\} : i \neq j \quad g \neq 0 \\
 A_I^{(1)}(N_1, N_0) \quad & g_a[\mathcal{D}_a, \mathcal{D}_i] = x_i \mathcal{D}_a \quad \forall i \in \{i\} \quad \forall a \in \{a\} \quad g_a \neq 0 \\
 & g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0.
 \end{aligned} \tag{4.1}$$

$$\begin{aligned}
 & g[\mathcal{D}_i, \mathcal{D}_j] = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad \forall i, j \in \{i\} : i \neq j \quad g \neq 0 \\
 & g_+ \mathcal{D}_i \mathcal{D}_b = -x_i \mathcal{D}_b \quad \forall (i \in \{i\}, b \in \{a\}) : i < b \quad g_+ \neq 0 \\
 A_I^{(2)}(N_1, N_0) \quad & g_- \mathcal{D}_a \mathcal{D}_i = x_i \mathcal{D}_a \quad \forall (i \in \{i\}, a \in \{a\}) : i > a \quad g_- \neq 0 \\
 & \text{where } g_- = -g_+ \quad \text{if there exist } \mathcal{D}_i, \mathcal{D}_a, \mathcal{D}_b : a < i < b \\
 & g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0.
 \end{aligned} \tag{4.2}$$

Here and in what follows we indicate two integers  $N_0$  and  $N_1$ —the numbers of indices in the sets  $\{i\}$  and  $\{a\}$ , respectively—in braces to specify the type of algebra. As the reader can notice looking closely at equation (4.2), specifying  $N_0$  and  $N_1$  one obtains several algebras all denoted by  $A_I^{(2)}(N_1, N_0)$ . We did not introduce a different notation for each algebra in order to simplify the notations. We adopted the same attitude also for other algebras described below (see equations (4.3) and (4.5)).

The algebra  $A_{II}(N_1)$  possesses a unique extension

$$\begin{aligned}
 & g_{ij} \mathcal{D}_i \mathcal{D}_j = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad \forall i, j \in \{i\} : i < j \\
 & g_{i+} \mathcal{D}_i \mathcal{D}_b = -x_i \mathcal{D}_b \quad \forall (i \in \{i\}, b \in \{a\}) : i < b \\
 & g_{i-} \mathcal{D}_a \mathcal{D}_i = x_i \mathcal{D}_a \quad \forall (i \in \{i\}, a \in \{a\}) : i > a. \\
 A_{II}(N_1, N_0) \quad & \text{Here } g_{ij} := g_i - g_j \quad g_{i+} := g_+ + g_i \quad g_{i-} := g_- - g_i \\
 & \text{where for all } i : g_i \neq -g_+ \quad g_i \neq g_- \quad g_i \neq g_j \quad \text{if } i \neq j \\
 & \text{and } g_- = -g_+ \quad \text{if there exists } \mathcal{D}_i, \mathcal{D}_a, \mathcal{D}_b : a < i < b. \\
 & g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0.
 \end{aligned} \tag{4.3}$$

The parameters  $g_+$  and  $g_-$  in algebras  $A_I^{(2)}(N_1, N_0)$  and  $A_{II}(N_1, N_0)$  remain independent

provided that mutual order of indices of the subsets  $\{i\}$  and  $\{a\}$  (and hence the order of the generators  $D_i$  and  $D_a$ , see (2.1)) is as follows:

$$i_1 < i_2 < \dots < i_k < a_1 < a_2 < \dots < a_{N_0} < i_{k+1} < i_{k+2} < \dots < i_{N_1}.$$

Only in this case all the  $B$ -type minimal subalgebras in the algebras (4.2) and (4.3) belong to the type  $B^{(2)}$ . In the presence of  $B^{(3)}$ , or  $B^{(4)}$  type triples the parameters  $g_+$  and  $g_-$  are constrained by the condition  $g_+ + g_- = 0$  (keep in mind that for stochastic processes all rates have to be non-negative).

*Algebras of type B.* We now consider the case where the set  $\{i\}$  contains exactly two indices ( $N_1 = 2$ ), say,  $i$  and  $j$ ,  $i < j$ . To obtain such algebras—we call them the algebras of type  $B$ —one should consider consistent extensions of the  $B$ -type triples (3.9), (3.10), (3.13) and (3.14) by  $(N_0 - 1)$  generators with their labels in the set  $\{a\}$ .

Starting with the  $B^{(1)}$ -type triple  $\{D_i, D_{a_1}, D_j\}$ , the only possibility is to add new generators  $D_{a_2}, D_{a_3}$ , etc such that all the minimal sub algebras  $\{D_i, D_{a_2}, D_j\}, \{D_i, D_{a_3}, D_j\}$ , etc satisfy again  $B^{(1)}$ -type relations (3.9). In this situation, we can arrange the alphabetic order of the generators as follows:  $D_j > D_{a_1} > D_{a_2} > \dots > D_{a_{N_0}} > D_i$  (cf with the comment below equation (3.14)). So it is natural to put in this case  $i = 0, j = N - 1$ . The extended algebra  $B^{(1)}(2, N_0)$  reads

$$\begin{aligned} g D_0 D_{N-1} - (g - \Lambda) D_{N-1} D_0 &= x_{N-1} D_0 - x_0 D_{N-1} & g \neq 0 \\ g_a D_0 D_a - (g_a - \Lambda) D_a D_0 &= -x_0 D_a \\ g_a D_a D_{N-1} - (g_a - \Lambda) D_{N-1} D_a &= x_{N-1} D_a & \forall 1 < a < N-1 \quad g_a \neq 0 \\ g_{ab} D_a D_b - g_{ba} D_b D_a &= 0 & \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0. \end{aligned} \tag{4.4}$$

The situation becomes different for the other  $B$ -type triples (3.10), (3.13) and (3.14). Extending these algebras one can get algebras containing all the  $B^{(2)}, B^{(3)}$  and  $B^{(4)}$ -type minimal subalgebras. Therefore we introduce a unified notation  $B^{(2)}(2, n_<, n, n_>)$  for extensions of the triples  $B^{(2)} - B^{(4)}$ . Here  $n_< + n + n_> = N_0$  and a mutual order of the indices  $i, j$  and the indices from the set  $\{a\}$  is as follows:

$$a_1 < \dots < a_{n_<} < i < a_{(n_<+1)} < \dots < a_{(n_<+n)} < j < a_{(n_<+n+1)} < \dots < a_{N_0}.$$

The algebra  $B^{(2)}(2, n_<, n, n_>)$  reads

$$\begin{aligned} g D_i D_j - (g - \Lambda) D_j D_i &= x_j D_i - x_i D_j & g \neq 0 \\ g_+ D_i D_a &= -x_i D_a & (g_+ - \Lambda) D_j D_a &= -x_j D_a \quad \forall a > j \\ (g_- - \Lambda) D_a D_i &= x_i D_a & g_- D_a D_j &= x_j D_a \quad \forall a < i \\ g_+ D_i D_a &= -x_i D_a & g_- D_a D_j &= x_j D_a & \forall i < a < j \end{aligned} \tag{4.5}$$

where  $g_+ \neq \begin{cases} 0 & \text{if } n_< < N_0 \\ \Lambda & \text{if } n_> > 0 \end{cases}$   $g_- \neq \begin{cases} 0 & \text{if } n_> < N_0 \\ \Lambda & \text{if } n_< > 0 \end{cases}$

and  $g_+ + g_- = \Lambda$  if among the numbers  $n_<, n, n_>$  there are two nonzeros

$$g_{ab} D_a D_b - g_{ba} D_b D_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0.$$

The algebra  $B^{(2)}(2, n_<, n, n_>)$  contains  $n_<$   $B^{(4)}$ -type minimal subalgebras ( $\{D_a, D_i, D_j\}$  for  $a < i$ ),  $n$   $B^{(2)}$ -type minimal subalgebras ( $\{D_i, D_a, D_j\}$  for  $i < a < j$ ) and  $n_>$   $B^{(3)}$ -type triples ( $\{D_i, D_j, D_a\}$  for  $a > j$ ).

*Algebras of type C.* Next, we consider algebras with only one index  $i$  in the subset  $\{i\}$ , let us call them the algebras of the type  $C$ . These algebras arise from extension of the  $C_i$  triple  $\{\mathcal{D}_i, \mathcal{D}_a, \mathcal{D}_b\}$  by  $(N_0 - 2)$  generators  $\mathcal{D}_c, \dots$  labelled by indices from the subset  $\{a\}$ . Checking the consistency of such an extension is straightforward and, therefore, we shall just present directly the resulting algebra

$$C(1, N_0) \quad \begin{aligned} &g_a \mathcal{D}_i \mathcal{D}_a - (g_a - \Lambda) \mathcal{D}_a \mathcal{D}_i = -x_i \mathcal{D}_a \\ &\text{where } g_a \neq 0 \quad \text{if } i < a \quad \text{and} \quad g_a \neq \Lambda \quad \text{if } a < i \quad (4.6) \\ &g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0. \end{aligned}$$

As in case of the  $C^{(1)}$ -type triple (3.15), for  $\Lambda \neq 0$  one can reduce the algebra (4.6) to a subcase of the family of quantum hyperplanes (see case  $D$  below) shifting the generator  $\mathcal{D}_i$ :  $\mathcal{D}_i \rightarrow \mathcal{D}_i - x_i/\Lambda$ . Nevertheless, as will be shown in the theorem given below, it is useful to keep the definition given by equation (4.6) for the algebra  $C(1, N_0)$  since in this way one can use it as a building block for the construction of new algebras. In the new algebras the shift will not be possible anymore.

Note that, unlike all the previous cases, one can consider the algebra  $C(1, 1)$  produced by a pair of generators. We shall use this possibility in the theorem stated below. For instance, the  $C^{(2)}$  algebra (3.16) can be constructed as a combination of two  $C(1, 1)$  algebras by a blending procedure described in the theorem given below. Further examples of an application of  $C(1, 1)$  algebras are given in appendix B (see the cases 17 and 18).

*Case D.* The  $D$  algebras—the algebras with no indices in the subset  $\{i\}$ —are represented by a family of  $N = N_0$ -dimensional quantum hyperplanes

$$D(0, N_0) \quad g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad \forall a, b \in \{a\} : a < b \quad g_{ab} \neq 0. \quad (4.7)$$

Now we are ready to complete a classification scheme. To do so one needs to take into consideration the possibility of using  $C^{(2)}$ -type triples in the algebra extension process. As a result one derives a procedure to obtain all the the diffusion algebras which is described in the following theorem.

**Theorem.** *A diffusion algebra has  $N_1$  generators  $\mathcal{D}_i$ , where one assumes  $x_i \neq 0$ , ( $i = 1, 2, \dots, N_1$ ) and  $N_0$  generators  $\mathcal{D}_a$ , with  $x_a = 0$ , ( $a = 1, 2, \dots, N_0$ ). If we do not distinguish between the two kinds of generators, we denote them by  $\mathcal{D}_\alpha$ , ( $\alpha = 1, 2, \dots, N = N_0 + N_1$ ).*

*If  $N_1 = 0$ , the algebras are  $D(0, N_0)$  (see equation (4.7)).*

*If  $N_1 \neq 0$ , all diffusion algebras can be obtained by a blending procedure using the algebras (4.1)–(4.6). The blending procedure can be described as follows.*

*Consider two of the algebras (4.1)–(4.6), denoted by  $X(N_1, N_0^{(x)})$  and  $Y(N_1, N_0^{(y)})$ , both having the same number of generators  $\mathcal{D}_i$  which satisfy the same relations among themselves in the two algebras, and generators  $\mathcal{D}_{a_x}$  (respectively  $\mathcal{D}_{a_y}$ ). Through blending, one can obtain a new diffusion algebra  $Z(N_1, N_0^{(x)} + N_0^{(y)} = N_0)$  with generators  $\mathcal{D}_i$  and  $\mathcal{D}_a$ , ( $a = 1, 2, \dots, N_0$ ). Since the  $X(N_1, N_0^{(x)})$  and  $Y(N_1, N_0^{(y)})$  algebras are both of PBW type, the  $N_0^{(x)}$  indices (respectively the  $N_0^{(y)}$  indices) are in a given alphabetic order. We blend now the  $N_0^{(x)}$  and  $N_0^{(y)}$  indices together in an arbitrary alphabetic order but respecting the order for the  $N_0^{(x)}$  indices (respectively the  $N_0^{(y)}$  indices) which are fixed in the algebra  $X(N_1, N_0^{(x)})$  (respectively  $Y(N_1, N_0^{(y)})$ ). The alphabetic order of the  $N_0^{(x)}$  indices in respect to the  $N_1$  indices (respectively the  $N_0^{(y)}$  indices in respect to the  $N_1$  indices) given again by the two algebras  $X(N_1, N_0^{(x)})$  and  $Y(N_1, N_0^{(y)})$ , has also to be respected. For each alphabetic order of the  $N_0$  indices one obtains a new algebra in the following way. The relations among the generators  $\mathcal{D}_i$  and those*

among the generators  $\mathcal{D}_i$  and  $\mathcal{D}_{a_x}$  (respectively among  $\mathcal{D}_i$  and  $\mathcal{D}_{a_y}$ ) coincide with the relations in the algebras  $X(N_1, N_0^{(x)})$  and  $Y(N_1, N_0^{(y)})$ . The remaining relations among the generators  $\mathcal{D}_{a_x}$  and  $\mathcal{D}_{a_y}$  are

$$\begin{aligned} \mathcal{D}_{a_x} \mathcal{D}_{a_y} &= 0 & \forall a_x, a_y : a_x < a_y \\ \mathcal{D}_{a_y} \mathcal{D}_{a_x} &= 0 & \forall a_x, a_y : a_x > a_y. \end{aligned} \tag{4.8}$$

A ‘blended’ algebra can now be blended with one of the algebras (4.1)–(4.6) and one can obtain a new algebra.

It is important to stress that through the blending procedure one can obtain the same algebra using different blendings. Therefore what we have is a construction rather than a classification of the diffusion algebras.

We would like to point out that it is easy to see that one can blend together only algebras of the same type:  $A_I^{(1)}$  or  $A_I^{(2)}$  with algebras  $A_I^{(1)}$  or  $A_I^{(2)}$ ,  $A_{II}$  with  $A_{II}$ ,  $B^{(1)}$  or  $B^{(2)}$  with  $B^{(1)}$  or  $B^{(2)}$  and  $C$  with  $C$ . One can show that blending together two  $A_I^{(1)}$  (respectively  $B^{(1)}$ ) algebras leads to  $A_I^{(1)}$  (respectively  $B^{(1)}$ ) algebras. Therefore one can use an  $A_I^{(1)}$  (respectively  $B^{(1)}$ ) algebra only once during blending procedure. At the same time the algebras  $A_I^{(2)}$  (respectively  $B^{(2)}$ ) can be blended any number of times.

Let us show an application of this theorem. Consider the algebra

$$g_{0a} \mathcal{D}_0 \mathcal{D}_a - g_{a0} \mathcal{D}_a \mathcal{D}_0 = -x_0 \mathcal{D}_a \quad a = 1, 2, \dots, N - 1 \tag{4.9}$$

$$g_{ab} \mathcal{D}_a \mathcal{D}_b = 0, \quad \forall a, b : 1 \leq a < b \leq N - 1. \tag{4.10}$$

This algebra is obtained taking  $N - 1$  copies of the algebra  $C(1, 1)$  ( $\{i\} = \{0\}$ ,  $\{a\} = \{1, 2, \dots, N - 1\}$ ) and blending them together. Taking  $N = 3$  and 4 one recovers the algebras given by equation (3.16) respectively equations (B.25), (B.26). If in equation (4.10) one takes the rates  $g_{ab} = 0$  one recovers the algebra discussed in [2].

### 5. Discussion

We have defined diffusion algebras. Those are PBW algebras with  $N$  generators satisfying the relations (1.11). These algebras are useful to find stationary states of the stochastic processes given by the rates  $g_{\alpha\beta}$ . For the  $N$  species problem one finds several series of algebras which might be useful in applications using equation (1.3) for a ring and equations (1.7) and (1.8) for open systems. Much work is still left. For example one has to find which boundary conditions, if any, are compatible with each of the algebras. We would like to stress that all the cases which were used up to now for applications are special cases of our construction.

An open and relevant question is: do non-PBW type ‘physically meaningful’ algebras satisfying equation (1.11) exist<sup>7</sup>. Such algebras could eventually represent ‘exceptional algebras’ similar to those which appear in the theory of simple Lie algebras or superalgebras.

An interesting and unsolved problem is the connection between the diffusion algebras and the so called quantum Lie algebras (see e.g. [18]) which look similar and which are not fully investigated and classified. In order to show that such a connection might be possible we will give here two examples of quantum Lie algebras related to the so-called Cremmer–Gervais  $R$  matrix [19]

$$\begin{aligned} g_\beta \mathcal{D}_\alpha \mathcal{D}_\beta - (g_\beta - \Lambda) \mathcal{D}_\beta \mathcal{D}_\alpha &= -x_\alpha \mathcal{D}_\beta \\ g \mathcal{D}_\alpha \mathcal{D}_\gamma - (g - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\alpha + w (\mathcal{D}_\beta)^2 &= x_\gamma \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\gamma \\ g_\beta \mathcal{D}_\beta \mathcal{D}_\gamma - (g_\beta - \Lambda) \mathcal{D}_\gamma \mathcal{D}_\beta &= x_\gamma \mathcal{D}_\beta \end{aligned} \tag{5.1}$$

<sup>7</sup> These algebras can be finite or infinite-dimensional.

and

$$\begin{aligned} g_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta &= -x_\alpha \mathcal{D}_\beta \\ g_{\alpha\gamma} \mathcal{D}_\alpha \mathcal{D}_\gamma - g_{\gamma\alpha} \mathcal{D}_\gamma \mathcal{D}_\alpha + w (\mathcal{D}_\beta)^2 &= x_\gamma \mathcal{D}_\alpha - x_\alpha \mathcal{D}_\gamma \\ g_{\beta\gamma} \mathcal{D}_\beta \mathcal{D}_\gamma &= x_\gamma \mathcal{D}_\beta. \end{aligned} \quad (5.2)$$

They are clearly extensions of the  $B^{(1)}$  and  $B^{(2)}$  algebras to which the term  $(\mathcal{D}_\beta)^2$  has been added ( $w$  is an arbitrary parameter). The algebras (5.1) and (5.2) are of PBW type. Quantum Lie algebras can also be relevant in a different context un-related to stochastic processes. As pointed out in [20], quadratic algebras are also useful to describe the ground states of one-dimensional quantum chains in equilibrium statistical physics if it happens that the ground states have energy zero. This is a whole area which is worth exploring.

Before closing this paper, we would like to mention a natural extension of our results. The starting point of our investigation were the processes given in equation (1.1) which are related to quantum hyperplanes (take all the  $x$  equal to zero in equation (1.11)). One can consider more general stochastic processes in which the bulk rates are related to quantum superplanes [16]. In these cases one obtains equations which generalize equation (1.11) (see [9]). This would lead us to something to which one could coin the name of ‘reaction-diffusion algebras’. For the time being this is no more than a nice thought.

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### Appendix A. Comments on diffusion algebras with three generators

We consider the relations (1.11) in the case when we have only three generators

$$\begin{aligned} g_{01} \mathcal{D}_0 \mathcal{D}_1 - g_{10} \mathcal{D}_1 \mathcal{D}_0 &= x_1 \mathcal{D}_0 - x_0 \mathcal{D}_1 \\ g_{20} \mathcal{D}_2 \mathcal{D}_0 - g_{02} \mathcal{D}_0 \mathcal{D}_2 &= x_0 \mathcal{D}_2 - x_2 \mathcal{D}_0 \\ g_{12} \mathcal{D}_1 \mathcal{D}_2 - g_{21} \mathcal{D}_2 \mathcal{D}_1 &= x_2 \mathcal{D}_1 - x_1 \mathcal{D}_2 \end{aligned} \quad (A.1)$$

and look for the case when the rates satisfy the condition

$$\Lambda_{012} = g_{01} - g_{10} + g_{12} - g_{21} + g_{20} - g_{02} = 0. \quad (A.2)$$

It is trivial to verify [12] that the  $1 \times 1$  matrices ( $c$ -numbers)

$$D_i = \frac{x_i}{f_i} \quad (i = 0, 1, 2) \quad (A.3)$$

where

$$f_1 = f_0 + g_{01} - g_{10} \quad f_2 = f_0 + g_{02} - g_{20} \quad (A.4)$$

verify the relations (A.1) ( $f_0$  is an arbitrary parameter). One can use (A.3) in order to compute, using equation (1.3), the probability distribution on a ring. The probability distribution one obtains for the stationary state is trivial (one has no correlations). The physics of the stationary state can however be interesting if one takes an open system. We then have to use equation (1.8)

and the one-dimensional representation of (A.1) is not of much help. If the rates satisfy only the condition (A.2) it is not clear if one does not have only the one-dimensional representation. At this point one can understand why we are interested in algebras of PBW type. For algebras of PBW type, we can be sure that one gets other representations (at least the regular one). The price to pay is that we will get more constraints on the rates than those given by equation (A.2). In order to get algebras of PBW type compatible with the relation (A.2), it is useful to write the diamond condition (2.4) in a different way:

$$\begin{aligned} \Lambda_{012} (x_0 g_{21} D_2 D_1 + x_1 g_{20} D_2 D_0 + x_2 g_{21} D_2 D_1 + x_1 x_2 D_0 - x_0 x_1 D_2) \\ + x_0 g_{12} g_{21} [D_1, D_2] + x_1 g_{02} g_{20} [D_2, D_0] \\ + x_2 g_{01} g_{10} [D_0, D_1] = 0. \end{aligned} \tag{A.5}$$

We now take into account equation (A.2). There are several solutions of f (A.5):

$$(a) \quad g_{ij} = g. \tag{A.6}$$

This gives (using equation (A.1)) the algebra  $A_I$  (see equation (3.5)):

$$(b) \quad \begin{aligned} g_{21} = g_{20} = g_{10} = 0 \\ g_{02} = g_{01} + g_{12}. \end{aligned} \tag{A.7}$$

This gives the algebra  $A_{II}$  (see equation (3.6)):

$$(c) \quad \begin{aligned} g_{20} = g_{10} = 0 \\ [D_1, D_2] = 0. \end{aligned} \tag{A.8}$$

This gives actually again the algebra  $A_{II}$ , defined in equation (3.6) with the substitution

$$g_{12} - g_{21} \longrightarrow g_{12}. \tag{A.9}$$

This different derivation of the algebra  $A_{II}$  has a bonus: we have learned that we can take the quotient given by the second relation in equation (A.8) of this algebra:

$$(d) \quad \begin{aligned} x_0 = 0 \\ g_{01} = g_{02} = 0 \\ g_{12} = g \quad g_{21} = g - \Lambda \\ g_{10} = g_\gamma \quad g_{20} = g_\gamma - \Lambda \end{aligned} \tag{A.10}$$

which is the algebra  $B^{(3)}$  (see equation (3.13)).

$$(e) \quad \begin{aligned} x_0 = 0 \\ g_{10} = g_{20} = 0 \\ g_{12} = g \quad g_{21} = g - \Lambda \\ g_{01} = g_\alpha - \Lambda \quad g_{02} = g_\alpha \end{aligned} \tag{A.11}$$

which is the algebra  $B^{(4)}$  (defined in equation (3.14)).

Since the algebras  $A_I$ ,  $A_{II}$ ,  $B^{(3)}$  and  $B^{(4)}$  are all derived starting from equation (A.1) with the conditions (A.2), all these algebras have at least a one-dimensional representation. When the classification of the relations (A.1) for which a trace operation exists was done [12], there was no need to consider them separately since for the trace operation it is enough to have the expressions (A.3) and (A.4).



## Appendix B. Diffusion algebras with positive rates for $N = 4$

Here we list all the diffusion algebras with  $N = 4$  generators which can be useful for stochastic processes with four species of particles (one can choose all the rates non-negative).

We first describe those of the algebras from the seven series described in section 4 (see equations (4.1)–(4.7)). For the reader's convenience we point out all the  $N = 3$  subalgebras for each algebra in the list. The reader may be surprised by the fact that in the list which follows the same symbol will denote two algebras (see equations (B.3) and (B.5)), this is due to the fact that as can be seen already in equations (4.2) and (4.3) the same notation is used for several algebras.

(1)  $A_I(4)$ :

$$g [\mathcal{D}_i, \mathcal{D}_j] = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad i, j = 0, 1, 2, 3. \quad (\text{B.1})$$

All the  $N = 3$  subalgebras of this algebra (i.e., the subalgebras generated by different triples of the generators  $\mathcal{D}_i, i = 0, 1, 2, 3$ ) are of type  $A_I$  (3.5).

(2)  $A_I^{(1)}(3, 1)$ :

$$\begin{aligned} g [\mathcal{D}_i, \mathcal{D}_j] &= x_j \mathcal{D}_i - x_i \mathcal{D}_j & i, j = 1, 2, 3 \\ g_o [\mathcal{D}_0, \mathcal{D}_i] &= x_i \mathcal{D}_0. \end{aligned} \quad (\text{B.2})$$

This algebra contains the  $A_I$  subalgebra  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$  and three  $B^{(1)}$  subalgebras (3.9) in which one takes  $\Lambda = 0$  and which are generated by the triples  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$ ,  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  and  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ .

(3)  $A_I^{(2)}(3, 1)$  (two algebras):

$$\begin{aligned} g [\mathcal{D}_i, \mathcal{D}_j] &= x_j \mathcal{D}_i - x_i \mathcal{D}_j & \text{and} & & g [\mathcal{D}_i, \mathcal{D}_j] &= x_j \mathcal{D}_i - x_i \mathcal{D}_j \\ g_o \mathcal{D}_0 \mathcal{D}_i &= x_i \mathcal{D}_0 & & & g_+ \mathcal{D}_1 \mathcal{D}_0 &= -x_1 \mathcal{D}_0 \\ & & & & g_- \mathcal{D}_0 \mathcal{D}_{2,3} &= x_{2,3} \mathcal{D}_0 \\ i, j &= 1, 2, 3. & & & i, j &= 1, 2, 3. \end{aligned} \quad (\text{B.3})$$

Both algebras presented in (B.3) contain the  $A_I$  triple  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$  and the  $B^{(4)}$  triple  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$  (see (3.14) specialized to  $\Lambda = 0$ ). The triples  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  and  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  are of  $B^{(4)}$  type for the first case in (B.3) and they belong to the type  $B^{(2)}$  for the last case (see (3.10) with the special choice  $g_{\alpha\gamma} = g_{\gamma\alpha} = g$ ).

There are two more algebras of type  $A_I^{(2)}(3, 1)$ . Their defining relations are obtained by inverting the order of the generators in all the products in formulae (B.3). This means that the last pair of algebras would describe physical processes which are just mirror reflections of the processes corresponding to the algebras (B.3).

(4)  $A_{II}(4)$ .

$$g_{ij} \mathcal{D}_i \mathcal{D}_j = x_j \mathcal{D}_i - x_i \mathcal{D}_j \quad \text{where } i, j = 0, 1, 2, 3 \quad \text{and} \quad i < j. \quad (\text{B.4})$$

Here  $g_{ij} := g_i - g_j$  and  $g_0 > g_1 > g_2 > g_3$  so that  $g_{ij} > 0$  for all  $i < j$ .

The algebra (B.4) contains only  $A_{II}$  subalgebras (3.6).

(5)  $A_{II}(3, 1)$  (two algebras):

$$\begin{aligned} g_{ij} \mathcal{D}_i \mathcal{D}_j &= x_j \mathcal{D}_i - x_i \mathcal{D}_j & \text{and} & & g_{ij} \mathcal{D}_i \mathcal{D}_j &= x_j \mathcal{D}_i - x_i \mathcal{D}_j \\ g_{0i} \mathcal{D}_0 \mathcal{D}_i &= x_i \mathcal{D}_0 & & & g_+ \mathcal{D}_1 \mathcal{D}_0 &= -x_1 \mathcal{D}_0 \\ & & & & g_{-2} \mathcal{D}_0 \mathcal{D}_2 &= x_2 \mathcal{D}_0 \\ & & & & g_{-3} \mathcal{D}_0 \mathcal{D}_3 &= x_3 \mathcal{D}_0. \end{aligned} \quad (\text{B.5})$$

Here both  $i$  and  $j$  take the values 1, 2, or 3; furthermore one has  $i < j$  in the first relations in (B.5);  $g_{ij} := g_i - g_j$ ,  $g_{0i} := g_0 - g_i$  and  $g_{-i} := g_- - g_i$ , where  $g_0 > g_1 > g_2 > g_3$  and  $g_- > g_2$ .

The algebras in (B.5) contain the  $A_{II}$  triple  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$  and the  $B^{(4)}$  triple  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$  (see (3.14) with  $g = \Lambda$ ). The triples  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  and  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  are of  $B^{(4)}$  type for the first case in (B.5) and they belong to type  $B^{(2)}$  for the last case (see (3.10) with  $g_{\gamma\alpha} = 0$ ). One can construct two more algebras which are mirror partners of those listed in (B.5) (cf the case 3).

(6)  $B^{(1)}(2, 2)$ .

$$\begin{aligned} g \mathcal{D}_0 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_0 &= x_3 \mathcal{D}_0 - x_0 \mathcal{D}_3 \\ g_a \mathcal{D}_0 \mathcal{D}_a - (g_a - \Lambda) \mathcal{D}_a \mathcal{D}_0 &= -x_0 \mathcal{D}_a \\ g_a \mathcal{D}_a \mathcal{D}_3 - (g_a - \Lambda) \mathcal{D}_3 \mathcal{D}_a &= x_3 \mathcal{D}_a \quad a = 1, 2 \\ g_{12} \mathcal{D}_1 \mathcal{D}_2 - g_{21} \mathcal{D}_2 \mathcal{D}_1 &= 0. \end{aligned} \tag{B.6}$$

This algebra contains two  $B^{(1)}$  triples (3.9):  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  and  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ , and two  $C^{(1)}$  triples (3.15):  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  and  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ .

(7)  $B^{(2)}(2, 0, 2, 0)$ .

$$\begin{aligned} g \mathcal{D}_0 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_0 &= x_3 \mathcal{D}_0 - x_0 \mathcal{D}_3 \\ g_+ \mathcal{D}_0 \mathcal{D}_a &= -x_0 \mathcal{D}_a \\ g_- \mathcal{D}_a \mathcal{D}_3 &= x_3 \mathcal{D}_a \quad a = 1, 2 \\ g_{12} \mathcal{D}_1 \mathcal{D}_2 - g_{21} \mathcal{D}_2 \mathcal{D}_1 &= 0. \end{aligned} \tag{B.7}$$

This algebra contains two  $B^{(2)}$  triples (3.10):  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  and  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ , and two  $C^{(1)}$  triples:  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  and  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ , where in equation (3.15) one takes  $g_\beta = g_\gamma = \Lambda$ , respectively  $g_\beta = g_\gamma = 0$ .

(8)  $B^{(2)}(2, 2, 0, 0)$ .

$$\begin{aligned} g \mathcal{D}_2 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_2 &= x_3 \mathcal{D}_2 - x_2 \mathcal{D}_3 \\ (h - \Lambda) \mathcal{D}_a \mathcal{D}_2 &= x_2 \mathcal{D}_a \\ h \mathcal{D}_a \mathcal{D}_3 &= x_3 \mathcal{D}_a \quad a = 0, 1 \\ g_{01} \mathcal{D}_0 \mathcal{D}_1 - g_{10} \mathcal{D}_1 \mathcal{D}_0 &= 0. \end{aligned} \tag{B.8}$$

This algebra contains two  $B^{(4)}$  triples (3.14):  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$  and  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ , and two  $C^{(1)}$  triples (3.15) with  $g_\beta = g_\gamma = 0$ :  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  and  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ .

The algebra  $B^{(2)}(2, 0, 0, 2)$  is a mirror partner of the algebra above.

(9)  $C(1, 3)$ :

$$\begin{aligned} g_a \mathcal{D}_0 \mathcal{D}_a - (g_a - \Lambda) \mathcal{D}_a \mathcal{D}_0 &= -x_0 \mathcal{D}_a \\ g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a &= 0 \quad a, b = 1, 2, 3. \end{aligned} \tag{B.9}$$

This algebra contains three  $C^{(1)}$  triples  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$ ,  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  and  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_2\}$ , and the triple  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$  of the type  $D$  (3.18).

For  $\Lambda \neq 0$ , one can do the shift  $\mathcal{D}_0 \rightarrow \mathcal{D}_0 - x_0/\Lambda$  and bring this algebra to the subcase of the family of quantum hyperplanes (B.10).

(10)  $D(0, 4)$  algebra, or quantum hyperplane:

$$g_{ab} \mathcal{D}_a \mathcal{D}_b - g_{ba} \mathcal{D}_b \mathcal{D}_a = 0 \quad a, b = 0, 1, 2, 3. \tag{B.10}$$

Obviously, all the triples here are of  $D$  type.

Next, we use the procedure of blending several diffusion algebras as described in the theorem at the end of section 4. There is no need to point out anymore the  $N = 3$

subalgebras for each example separately, since the pair of main  $N = 3$  constituents which are blended to produce an  $N = 4$  algebra are mentioned explicitly in each case. The remaining two  $N = 3$  subalgebras always belong to the type  $C^{(2)}$  (3.16).

Note that blending algebras of the types  $A_I$  and  $A_{II}$  produces only examples with  $N \geq 5$ . One can get  $N = 4$  diffusion algebras by blending any two of the following  $N = 3$  type  $B$  algebras:  $B^{(1)}(2, 1) (\equiv B^{(1)})$  in the notations of section 3),  $B^{(2)}(2, 1, 0, 0) (\equiv B^{(4)})$ ,  $B^{(2)}(2, 0, 1, 0) (\equiv B^{(2)})$ , and  $B^{(2)}(2, 0, 0, 1) (\equiv B^{(3)})$ . The results are listed below.

- (11) Gluing  $B^{(1)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$ ) and  $B^{(2)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_0 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_0 &= x_3 \mathcal{D}_0 - x_0 \mathcal{D}_3 \\ g_1 \mathcal{D}_0 \mathcal{D}_1 - (g_1 - \Lambda) \mathcal{D}_1 \mathcal{D}_0 &= -x_0 \mathcal{D}_1 \\ g_1 \mathcal{D}_1 \mathcal{D}_3 - (g_1 - \Lambda) \mathcal{D}_3 \mathcal{D}_1 &= x_3 \mathcal{D}_1 \\ g_+ \mathcal{D}_0 \mathcal{D}_2 = -x_0 \mathcal{D}_2 \quad g_- \mathcal{D}_2 \mathcal{D}_3 &= x_3 \mathcal{D}_2. \end{aligned} \quad (\text{B.11})$$

This set of relations should be supplemented by the condition

$$g_{12} \mathcal{D}_1 \mathcal{D}_2 = 0. \quad (\text{B.12})$$

The algebra (B.11), (B.12) has a mirror partner with an opposite order of indices 1 and 2 (one uses the condition  $g_{21} \mathcal{D}_2 \mathcal{D}_1 = 0$  instead of (B.12) for it).

- (12) Gluing  $B^{(2)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$ ) and  $B^{(2)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_0 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_0 &= x_3 \mathcal{D}_0 - x_0 \mathcal{D}_3 \\ g_+ \mathcal{D}_0 \mathcal{D}_1 = -x_0 \mathcal{D}_1 \quad g_- \mathcal{D}_1 \mathcal{D}_3 &= x_3 \mathcal{D}_1 \\ h_+ \mathcal{D}_0 \mathcal{D}_2 = -x_0 \mathcal{D}_2 \quad h_- \mathcal{D}_2 \mathcal{D}_3 &= x_3 \mathcal{D}_2 \end{aligned} \quad (\text{B.13})$$

and

$$g_{12} \mathcal{D}_1 \mathcal{D}_2 = 0. \quad (\text{B.14})$$

- (13) Gluing  $B^{(1)}$  (with generators  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ ) and  $B^{(4)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_2 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_2 &= x_3 \mathcal{D}_2 - x_2 \mathcal{D}_3 \\ (h - \Lambda) \mathcal{D}_0 \mathcal{D}_2 = x_2 \mathcal{D}_0 \quad h \mathcal{D}_0 \mathcal{D}_3 &= x_3 \mathcal{D}_0 \\ (g_1 - \Lambda) \mathcal{D}_1 \mathcal{D}_2 - g_1 \mathcal{D}_2 \mathcal{D}_1 &= x_2 \mathcal{D}_1 \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} g_1 \mathcal{D}_1 \mathcal{D}_3 - (g_1 - \Lambda) \mathcal{D}_3 \mathcal{D}_1 &= x_3 \mathcal{D}_1 \\ \text{and either } g_{01} \mathcal{D}_0 \mathcal{D}_1 = 0 \quad \text{or } g_{10} \mathcal{D}_1 \mathcal{D}_0 &= 0. \end{aligned} \quad (\text{B.16})$$

Gluing the algebras  $B^{(1)}$  and  $B^{(3)}$  produces a mirror partner of this algebra.

- (14) Gluing  $B^{(4)}$  (with generators  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ ) and  $B^{(4)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_2 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_2 &= x_3 \mathcal{D}_2 - x_2 \mathcal{D}_3 \\ (h - \Lambda) \mathcal{D}_0 \mathcal{D}_2 = x_2 \mathcal{D}_0 \quad h \mathcal{D}_0 \mathcal{D}_3 &= x_3 \mathcal{D}_0 \\ (f - \Lambda) \mathcal{D}_1 \mathcal{D}_2 = x_2 \mathcal{D}_1 \quad f \mathcal{D}_1 \mathcal{D}_3 &= x_3 \mathcal{D}_1 \end{aligned} \quad (\text{B.17})$$

and

$$g_{01} \mathcal{D}_0 \mathcal{D}_1 = 0. \quad (\text{B.18})$$

Gluing the algebras  $B^{(3)}$  and  $B^{(3)}$  produces a mirror partner of this algebra.

- (15) Gluing  $B^{(3)}$  (with generators  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ ) and  $B^{(4)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_1 \mathcal{D}_2 - (g - \Lambda) \mathcal{D}_2 \mathcal{D}_1 &= x_2 \mathcal{D}_1 - x_1 \mathcal{D}_2 \\ h \mathcal{D}_1 \mathcal{D}_3 = -x_1 \mathcal{D}_3 \quad (h - \Lambda) \mathcal{D}_2 \mathcal{D}_3 &= -x_2 \mathcal{D}_3 \\ (f - \Lambda) \mathcal{D}_0 \mathcal{D}_1 = x_1 \mathcal{D}_0 \quad f \mathcal{D}_0 \mathcal{D}_2 &= x_2 \mathcal{D}_0 \end{aligned} \quad (\text{B.19})$$

and

$$g_{03} \mathcal{D}_0 \mathcal{D}_3 = 0. \quad (\text{B.20})$$

- (16) Gluing  $B^{(2)}$  (with generators  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ ) and  $B^{(4)}$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g \mathcal{D}_1 \mathcal{D}_3 - (g - \Lambda) \mathcal{D}_3 \mathcal{D}_1 &= x_3 \mathcal{D}_1 - x_1 \mathcal{D}_3 \\ g_+ \mathcal{D}_1 \mathcal{D}_2 = -x_1 \mathcal{D}_2 \quad g_- \mathcal{D}_2 \mathcal{D}_3 &= x_3 \mathcal{D}_2 \\ (h - \Lambda) \mathcal{D}_0 \mathcal{D}_1 = x_1 \mathcal{D}_0 \quad h \mathcal{D}_0 \mathcal{D}_3 &= x_3 \mathcal{D}_0 \end{aligned} \quad (\text{B.21})$$

and

$$g_{02} \mathcal{D}_0 \mathcal{D}_2 = 0. \quad (\text{B.22})$$

Gluing the algebras  $B^{(2)}$  and  $B^{(3)}$  produces mirror partner of this algebra.

Gluing a pair of  $B^{(1)}$  algebras gives a special case of the  $B^{(1)}(2, 2)$  algebra (see case 6 above) with  $g_{21} = 0$ .

There are two other possibilities to blend the  $C$  type algebras  $C(1, 2)$  and  $C(1, 1)$  (for their definition see equation (4.6)) into a  $N = 4$  diffusion algebra.

- (17) Gluing  $C(1, 2)$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$ ) and  $C(1, 1)$  (with generators  $\{\mathcal{D}_0, \mathcal{D}_3\}$ ) one obtains

$$\begin{aligned} g_a \mathcal{D}_0 \mathcal{D}_a - (g_a - \Lambda) \mathcal{D}_a \mathcal{D}_0 &= -x_0 \mathcal{D}_a \quad a = 1, 2 \\ g_{12} \mathcal{D}_1 \mathcal{D}_2 - g_{21} \mathcal{D}_2 \mathcal{D}_1 &= 0 \\ g_{03} \mathcal{D}_0 \mathcal{D}_3 - g_{30} \mathcal{D}_3 \mathcal{D}_0 &= -x_0 \mathcal{D}_3 \end{aligned} \quad (\text{B.23})$$

with either one of the following two sets of conditions

$$g_{13} \mathcal{D}_1 \mathcal{D}_3 = g_{23} \mathcal{D}_2 \mathcal{D}_3 = 0 \quad \text{or} \quad g_{13} \mathcal{D}_1 \mathcal{D}_3 = g_{32} \mathcal{D}_3 \mathcal{D}_2 = 0. \quad (\text{B.24})$$

In this algebra, besides the  $C^{(1)}$  triple  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\}$  there are two  $C^{(2)}$  triples  $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_3\}$  and  $\{\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_3\}$  and the  $D$  triple  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ .

- (18) Gluing three copies of  $C(1, 1)$  algebra one obtains

$$g_{0a} \mathcal{D}_0 \mathcal{D}_a - g_{a0} \mathcal{D}_a \mathcal{D}_0 = -x_0 \mathcal{D}_a \quad a = 1, 2, 3 \quad (\text{B.25})$$

and

$$g_{12} \mathcal{D}_1 \mathcal{D}_2 = g_{13} \mathcal{D}_1 \mathcal{D}_3 = g_{23} \mathcal{D}_2 \mathcal{D}_3 = 0. \quad (\text{B.26})$$

This algebra contains one  $D$  type triple  $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$  and all the other triples are of the type  $C^{(2)}$ .

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